Derivatives of Multivariable Functions

Idea! The derivative measures change in output for corresponding small change in input... In some small direction

Def^n: Let f be a function of n-variables and $\vec{u}$ a unit vector in $\mathbb{R}^n$. Let $\vec{a} \in \operatorname{dom}(f)$. The direction derivative of f at $\vec{a}$ in direction of $\vec{u}$ is $D_{\vec{u}} f(\vec{a}) = \lim_{h \to 0} \frac{f(\vec{a}+h\vec{u}) - f(\vec{a})}{h}$

Ex. Comp dir deriv of $f(x,y) = xy$ at $\vec{a} = \langle 1, 3 \rangle$ in direction $\vec{u} = \frac{1}{2}\langle \sqrt{3}, \sqrt{2} \rangle$

Sol: $D_{\vec{u}} f(\vec{a}) = \lim_{h \to 0^+} \frac{f(\vec{a}+h\vec{u}) - f(\vec{a})}{h} = \lim_{h \to 0^+} \frac{f(1+\frac{\sqrt{2}}{2}h, 3+\frac{\sqrt{2}}{2}h) - f(1,3)}{h}$

$\lim_{h \to 0^+} \frac{(1+\frac{\sqrt{2}}{2})(3+\frac{\sqrt{2}}{2}h) - 1\cdot 3}{h} = \lim_{h \to 0^+} \frac{(3 + h(\frac{3\sqrt{2}}{2}+\frac{1\sqrt{3}}{2}) + h^2) - 3}{h} = \lim_{h \to 0^+} \frac{h(2\sqrt{2}+h)}{h}$

$= \lim_{h \to 0^+} (2\sqrt{3}+h) = 2\sqrt{2}+0 = 2\sqrt{2}$

Exercise: Find gen formula by substituting $\vec{a} = \langle x, y \rangle$

NB: The dir deriv. is very general.
We want something like "rules" from Calc I...

Def^n: Let f be a function of n-variables and let $\vec{e}_k$ be k-th standard basis vector in $\mathbb{R}^n$, i.e. $e_k = \langle 0, 0, \dots, \underset{k^{th}\text{ position}}{1}, \dots, 0 \rangle$

The k^{th} partial derivative of f (alt. the partial deriv of k^{th} pos. f with respect to x_k) is $D_{\vec{e}_k} f(\vec{a})$

Starting here

$\frac{\partial f}{\partial k_n} = D_{\vec{e}_k} f$ ← k^{th} notation for k^{th} pos. deriv

$e_1 = \langle 1, 0 \rangle$

What's going on here?
Let's think about $n=2$: $f(x,y)$

$\frac{\partial f}{\partial x}\Big|_{(a,b)} = D_{\vec{e}_1} f(\overset{x_1}{a}, \overset{x_2}{b})$

$= \lim_{h \to 0^+} \frac{f(\langle a,b \rangle + h e_1) - f(a,b)}{h}$

$= \lim_{h \to 0^+} \frac{f(a+h, b) - f(a,b)}{h}$

Define $g(x) = f(x,b)$. This line becomes:

$\frac{\partial f}{\partial x}\Big|_{(a,b)} = \lim_{h \to 0^+} \frac{g(a+h) - g(a)}{h}$ ← usual derivative all usual properties hold...

$= g'(a)$ ← by calc I

Ex. $f(x,y) = xy$
$b = \pi$
$g(x) = f(x, \pi)$
$= x\pi$

Point: $\frac{\partial f}{\partial x}$ is the "usual deriv." of f, pretending that every variable except for x is constant! (that was the point of g...)

– Similarly $\frac{\partial f}{\partial y}$ is the deriv of f holding x constant...

curly ∂ represents multiple variables

Ex. Consider the partial derivatives of $f(x,y) = xy + \sqrt{y} - \sin(x-y)$

Sol: $\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}[xy + \sqrt{y} - \sin(x-y)] = \frac{\partial}{\partial x}[xy] + \frac{\partial}{\partial x}[\sqrt{y}] - \frac{\partial}{\partial x}[\sin(x-y)]$ chain rule

use single variable deriv. properties

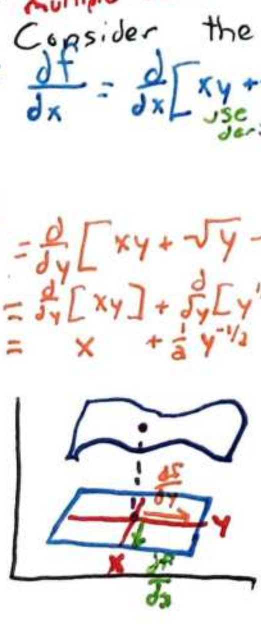
$= y\frac{\partial}{\partial x}[x] + \underset{\text{is constant w.r.t. } x}{0} - \cos(x-y)\frac{\partial}{\partial x}[x-y]$

$= y(1) - \cos(x-y)(1) = y - \cos(x-y)$

$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}[xy + \sqrt{y} - \sin(x-y)]$

$= \frac{\partial}{\partial y}[xy] + \frac{\partial}{\partial y}[y^{1/2}] - \frac{\partial}{\partial y}[\sin(x-y)]$

$= x + \frac{1}{2}y^{-1/2} - \cos(x-y)\frac{\partial}{\partial y}[x-y] = x + \frac{1}{2y^{1/2}} + \cos(x-y)$



Derivatives of Multivariable Functions cont.

Ex. Comp. partial deriv. of $f(x,y,z) = e^{x^2+y^2}\sin(xz)\cos(yz)$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left[e^{x^2+y^2}\sin(xz)\cos(yz)\right] = \cos(yz)\frac{\partial}{\partial x}\left[e^{x^2+y^2}\sin(xz)\right]$$

$$= \cos(yz)\left(\frac{\partial}{\partial x}\left[e^{x^2+y^2}\right]\sin(xz) + \frac{\partial}{\partial x}\left[\sin(xz)\right]e^{x^2+y^2}\right)$$

$$= \cos(yz)\left(2xe^{x^2+y^2}\sin(xz) + e^{x^2+y^2}z\cos xz\right)$$

$$= e^{x^2+y^2}\cos(yz)\left(2x\sin(xz) + z\cos(xz)\right)$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left[e^{x^2+y^2}\sin(xz)\cos(yz)\right] = \sin(xz)\frac{\partial}{\partial y}\left[e^{x^2+y^2}\cos(yz)\right]$$

$$= \sin(xz)\left(\frac{\partial}{\partial x}\left[e^{x^2+y^2}\right]\cos(yz) + \frac{\partial}{\partial x}\left[\cos(yz)\right]e^{x^2+y^2}\right)$$

$$= \sin(xz)\left(2ye^{x^2+y^2}\cos(yz) + e^{x^2+y^2}(-z\sin(yz))\right)$$

$$= \sin(xz)e^{x^2+y^2}\left(2y\cos(yz) - z\sin(yz)\right)$$

$$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}\left[e^{x^2+y^2}\sin(xz)\cos(yz)\right] = e^{x^2+y^2}(\sin(xz)\cos(yz))$$

$$= e^{x^2+y^2}\left(\frac{\partial}{\partial z}\left[\sin(xz)\right]\cos(yz) + \frac{\partial}{\partial z}\left[\cos(yz)\right]\sin(xz)\right)$$

$$= e^{x^2+y^2}\left(x\cos(xz)\cos(yz) + (y(-\sin(yz))\sin(xz)\right)$$

$$= e^{x^2+y^2}\left(x\cos(xz)\cos(yz) - y\sin(yz)\sin(xz)\right)$$

NB: higher order partial deriv. still make sense just like Calc I <u>except</u> now there's more

→ ... If $f(x,y)$ is given, the second order partials are:

$$\underbrace{\frac{\partial^2 f}{(\partial x)^2}, \frac{\partial^2 f}{(\partial y)^2}}_{\text{"pure partial deriv"}}, \underbrace{\frac{\partial^2 f}{\partial y\partial x}, \frac{\partial^2 f}{\partial x\partial y}}_{\text{mixed partial deriv}}$$

$\frac{\partial^2 f}{\partial y\partial x}$ ← deriv of x first then y

$\frac{\partial^2 f}{\partial x\partial y}$ ← deriv of y first then x

Ex. Comp. 2nd-order partial deriv. of $f(x,y) = xy + \sqrt{y} - \sin(x-y)$

Earlier we computed:

$$\frac{\partial f}{\partial x} = y - \cos(x-y) \quad \& \quad \frac{\partial f}{\partial y} = x + \frac{1}{2}y^{-1/2} + \cos(x-y)$$

$$\frac{\partial^2 f}{(\partial x)^2} = \frac{\partial}{\partial x}\left[y - \cos(x-y)\right] = \sin(x-y)$$

$$\frac{\partial^2 f}{(\partial y)^2} = \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial y}\right] = \frac{\partial}{\partial y}\left[x + \frac{1}{2}y^{-1/2} + \cos(x-y)\right] = -\frac{1}{4}y^{-3/2} + \sin(x-y)$$

$$\frac{\partial^2 f}{\partial y\partial x} = \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right] = \frac{\partial}{\partial y}\left[y - \cos(x-y)\right] = 1 - \sin(x-y)$$

$$\frac{\partial^2 f}{\partial x\partial y} = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right] = \frac{\partial}{\partial x}\left[x + \frac{1}{2}y^{-1/2} + \cos(x-y)\right] = 1 - \sin(x-y)$$

Interlude: These are truly just calc I deriv. ...
Working w/ 1 variable at a time allows us to utilize calc I strategies!

Deriv of Multivariable Functions cont.

Backed to mixed partials (somehow using both variables)

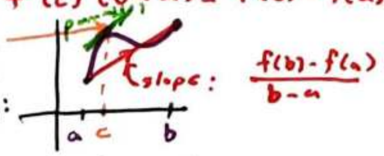
1. Why were these equal in our example?
2. Can we garentee this in future examples

Recall some Calc I, ~~Mean~~ Nice Average Value Thm...

Prop (Mean Value Thm): Let $f(t)$ be a function differentiable on (a,b) and cnts on $[a,b]$.

There is a value $a<c<b$ such that $f'(c)(b-a) = f(b) - f(a)$

Idea: There is a pt, c, in (a,b) so that



Next time, we use MVT to prove the following:

Prop (Clairaut's Thm):

Suppose $f(x,y)$ has cont. second-order partial deriv. on some disk including pt. (a,b)

Then $\left.\frac{\partial^2 f}{\partial y \partial x}\right|_{(a,b)} = \left.\frac{\partial^2 f}{\partial x \partial y}\right|_{(a,b)}$